Taylor-Görtler Instability of Compressible Boundary Layers

Ryōji Kobayashi* and Yasuaki Kohama†
Tōhoku University, Sendai, Japan

The purpose of this paper is to consider analytically how compressibility of a fluid will affect the instability of laminar boundary layers along slightly concave walls leading to the onset of longitudinal vortices. Neutral stability curves representing the relation of the Görtler parameter to the dimensionless wavenumber of the longitudinal vortices and also distributions of the disturbances at the onset of the vortices are presented. The results show that the critical value of the Görtler parameter increases about 1.6 times as the freestream Mach number varies from 0 to 5 in the case of a thermally insulated wall. Effects of temperature ratio (wall temperature to freestream temperature) in cases of an isothermal wall are also discussed.

Nomenclature

G	$= Re\sqrt{\theta/R}$, Görtler parameter
G_0	= Görtler parameter at $Ma_\infty = 0$
L	$= \lambda_0/\lambda_\infty$
M	$= \mu_0/\mu_\infty$
Ma_∞	= freestream Mach number
N	$= \rho_0/\rho_\infty$
Pr	= Prandtl number
p	= pressure
R	= radius of curvature on streamline
Re	$= \rho_\infty U_\infty \theta/\mu_\infty$, Reyonlds number based on U_∞ and θ
T	= temperature
$\bar{T}$	$= T_0/T_\infty$
t	= time
U_0	= streamwise component of base flow velocity in laminar boundary layer
U_∞	= freestream velocity
$\bar{U}$	$= U_0/U_\infty$
u, v, w	= velocity components in x, y, and z direction, respectively
$\bar{u}$	$= \hat{u}/U_\infty$
$\bar{v}$	$= Re\ \hat{v}/U_\infty$
$\bar{w}$	$= Re\ \hat{w}/U_\infty$
α	= wavenumber of longitudinal vortices
β	= growth rate of perturbation
ϵ	= increment of the Görtler parameter $[(G-G_0)/G_0]$
θ	= momentum thickness of laminar boundary layer
κ	= ratio of specific heats
λ	= thermal conductivity of gas
μ	= gas viscosity
η	$= y/\theta$
ρ	= gas density
σ	$= \alpha\theta$, dimensionless wavenumber
τ	$= \hat{T}/T_\infty$

Superscripts and Subscripts

$'$	= derivative with respect to η
$\hat{}$	= amplitude of perturbation
0	= base quantity in laminar boundary layer
w	= solid wall condition
∞	= freestream condition

Introduction

IN the transition problem of laminar boundary layers along concave curved surfaces, the Taylor-Görtler instability for three-dimensional disturbances of the longitudinal vortex type is called in question, as is the Tollmien-Schlichting instability for two-dimensional wavy disturbances. The former instability was first successfully investigated by Görtler,[1] and has hence been discussed mainly in incompressible flows; that is, the problem was treated more accurately,[2,3] and the role of a velocity component normal to a concave wall[4-6] as well as the effect of heat transfer[7] were studied. It was also extended to rarefied gas flow,[8] MHD boundary layers,[9-11] and turbulent boundary layers.[12-15] Liepmann[16] found experimentally that the Görtler parameter G is the proper critical parameter governing boundary layer instability due to concave curvature. The linear stability theories mentioned previously were supported by experiments.[17-19]

The effect of gas compressibility on the present instability was studied by Aihara[20] and Hämmerlin.[21] The former analysis[20] is made for two extreme cases of the Mach number, $Ma_\infty \ll 1$ and $Ma_\infty \gg 1$, in which the viscosity μ and the thermal conductivity λ are assumed to be constant. The latter treatment[21] is also restricted to $Ma_\infty \ll 1$, and assumes $\mu \propto T^{0.78}$. However, these two previous treatments gave opposite tendencies with respect to the effect of compressibility. Zakkay and Calarese[22] observed the longitudinal vortices at the freestream Mach number 5.75.

In the present paper, we consider the onset of longitudinal vortices in compressible boundary layers along slightly concave walls over a wide range of Ma_∞, where, in order to assure better accuracy, Sutherland's formula is adopted for the temperature dependence of the viscosity. The roles of the density gradient and the viscosity distribution in the instability of the boundary layer are discussed.

Perturbation Equations

A curvilinear coordinate system (x,y,z) is used, as shown in Fig. 1, where x is the arc length of the undisturbed concave streamlines, y is the distance measured perpendicular to the concave wall, and z runs along the wall and is normal to the other two axes. A uniform flow with a velocity U_∞ is directed along the x axis. For incompressible flow, Smith[4] considered the effect of boundary layer growth on the present instability, and Kobayashi[5,6] examined the role of vertical velocity component (V_0). In the present analysis the vertical component V_0 is neglected for simplicity. We assume that a radius of curvature R on the concave streamlines remains constant in the x direction and is large compared with the momentum thickness θ of the undisturbed boundary layer. Although in supersonic flows the pressure along concave walls tends to depend on $Ma_\infty x/R$, the principal effect of the pressure gradient on the stability characteristics of the boundary layer would be modifications of the base velocity profile $U_0(y)$ in the perturbation equations.

Received Oct. 19, 1976; revision received Aug. 26, 1977.
Index category: Boundary-Layer Stability and Transition.
*Professor, Institute of High Speed Mechanics.
†Research Assistant, Institute of High Speed Mechanics.

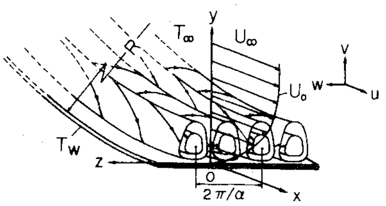

Fig. 1 Longitudinal vortices and coordinate system.

We shall now suppose that a two-dimensional compressible laminar boundary layer along the concave wall is slightly disturbed with the type of longitudinal vortices. Physical quantities in this disturbed flow are expressed as follows:

$$
\begin{aligned}
u &= U_0(y) + \hat{u}(y)e^{\beta t}\cos\alpha z, & v &= \hat{v}(y)e^{\beta t}\cos\alpha z \\
w &= \hat{w}(y)e^{\beta t}\sin\alpha z, & p &= p_0(y) + \hat{p}(y)e^{\beta t}\cos\alpha z \\
T &= T_0(y) + \hat{T}(y)e^{\beta t}\cos\alpha z, & \rho &= \rho_0(y) + \hat{\rho}(y)e^{\beta t}\cos\alpha z \\
\mu &= \mu_0(y) + \hat{\mu}(y)e^{\beta t}\cos\alpha z, & \lambda &= \lambda_0(y) + \hat{\lambda}(y)e^{\beta t}\cos\alpha z
\end{aligned} \tag{1}
$$

Because a wide range of the temperature is treated in the analysis, we adopt Sutherland's formula for the temperature dependence of viscosity:

$$\mu(T) = \mu_\infty \left(\frac{T}{T_\infty}\right)^{3/2} \frac{T_\infty + 110.6}{T + 110.6} \tag{2}$$

Prandtl number Pr and specific heat C_p are assumed to be constant, so that the thermal conductivity λ is obtained from the relation $\lambda(T) = \mu(T)C_p/Pr$. Substituting Eqs. (1) into a system of basic equations of momentum, energy, continuity, and state, which were written in the present curvilinear coordinate system; linearizing the obtained equations with respect to the disturbed quantities; and eliminating $\hat{p}$, $\hat{\rho}$, $\hat{\lambda}$, and $\hat{\mu}$ with relations $\hat{\lambda} = (d\lambda/dT)_0\hat{T}$ and $\hat{\mu} = (d\mu/dT)_0\hat{T}$; we reach finally the following perturbation equations for neutral stability ($\beta = 0$) under the condition $Ma_\infty^2\theta/R \ll 1$, which is used for a condition $\rho_0 U_0^2 \hat{p}/(Rp_0) \ll \partial\hat{p}/\partial y$:

$$\bar{u}'' + (ln\ M)'\bar{u}' - \sigma^2\bar{u} = \frac{N\bar{U}'}{M}\bar{v} - \frac{1}{M}\left(\bar{U}'\frac{dM}{d\bar{T}}\tau\right)' \tag{3}$$

$$\bar{v}'''' + B_2(\eta)\bar{v}''' + C_2(\eta)\bar{v}'' + D_2(\eta)\bar{v}' + E_2(\eta)\bar{v} = F_2(\eta) \tag{4}$$

$$\tau'' + 2(ln\ L)'\tau' + C_3(\eta)\tau = F_3(\eta) \tag{5}$$

$$\bar{w} = -1/\sigma N(N\bar{v})' \tag{6}$$

where the primes denote differentiation with respect to η and

$$B_2(\eta) = 2(ln\ M)' + (ln\ N)' \tag{7a}$$

$$C_2(\eta) = \frac{M''}{M} + \frac{2M'(ln\ N)'}{M} + 3(ln\ N)'' - 2\sigma^2 \tag{7b}$$

$$D_2(\eta) = \frac{M''(ln\ N)'}{M} + (ln\ M)'\{4(ln\ N)'' - 2\sigma^2\} + 3(ln\ N)''' - \sigma^2(ln\ N)' \tag{7c}$$

$$E_2(\eta) = \frac{M''}{M}\{(ln\ N)'' + \sigma^2\} + (ln\ M)'\{2(ln\ N)''' - 2\sigma^2(ln\ N)'\} + (ln\ N)'''' - \sigma^2(ln\ N)'' + \sigma^4 \tag{7d}$$

$$F_2(\eta) = \sigma^2 G^2 \frac{N\bar{U}}{M}\left(-2\bar{u} + \frac{\bar{U}}{\bar{T}}\tau\right) \tag{7e}$$

$$C_3(\eta) = -\sigma^2 + \frac{L''}{L} + Pr(\kappa - 1)Ma_\infty^2 \frac{(\bar{U}')^2}{L}\frac{dM}{d\bar{T}} \tag{7f}$$

$$F_3(\eta) = \frac{PrNT'}{L}\bar{v} - 2Pr(\kappa - 1)Ma_\infty^2 \frac{M\bar{U}'}{L}\bar{u}' \tag{7g}$$

As a limiting case of $Ma_\infty \rightarrow 0$, the system of the perturbation Eqs. (3-6) reduces to the previous ones.[1]

The boundary conditions which arise from the requirement of no slip at the wall surface ($\eta = 0$) are $\bar{u} = \bar{v} = \bar{v}' = 0$ in view of continuity equation (6). For the purpose of examining the effects of the thermal boundary condition, we consider two extreme cases; that is, $\tau' = 0$ for an ideally insulated wall, and $\tau = 0$ for completely conducting wall. As for the other four boundary conditions, it is reasonable to take $\bar{u} = \bar{v} = \bar{v}' = \tau = 0$ not at the outer edge of the boundary layer but at a point ($\eta \rightarrow \infty$) far from the wall, because the longitudinal vortices grow beyond the boundary layer. For convenience of numerical analysis we use, instead of the above boundary conditions as $\eta \rightarrow \infty$, the following four identities (8-11) available everywhere in the outer region of the boundary layer. With the conditions $\bar{U}' = 0$ and $\bar{T}' = 0$, Eq. (3) gives a solution $\bar{u} = A\ \exp(-\sigma\eta)$ (A = integration constant), which satisfies the relation

$$\bar{u}' + \sigma\bar{u} = 0 \tag{8}$$

Similarly, Eqs. (4) and (5) give the relations

$$\tau' + \sigma\tau = 0 \tag{9}$$

$$\bar{v}'' + 2\sigma\bar{v}' + \sigma^2\bar{v} = \frac{G^2}{4}(-2\bar{u} + \tau) \tag{10}$$

Differentiating Eq. (10) with respect to η and using Eqs. (8) and (9), we obtain the fourth identity

$$\bar{v}''' + 2\sigma\bar{v}'' + \sigma^2\bar{v}' = -\sigma G^2/4(-2\bar{u} + \tau) \tag{11}$$

Results and Discussions

The simultaneous differential equations (3-5) were numerically solved with an iterative procedure, and G was found as an eigenvalue together with the perturbation amplitudes of the velocities $\bar{u}(\eta)$, $\bar{v}(\eta)$ and the temperature $\tau(\eta)$ as eigenfunctions. In the numerical calculations by finite difference, the flow field ($\eta = 0$-18) was divided into 360 segments, and Eqs. (3) to (5) were solved by the forward elimination, backward substitution technique, where the solutions satisfied the boundary conditions at $\eta = 0$ and Eqs. (8-11) at $\eta = 18$. $\bar{w}(\eta)$ is then obtained from Eq. (6). The base velocity profile was approximately expressed by the one for a flat plate, which is typical and well defined. With a stream function $\psi_0 = (2\rho_\infty\mu_\infty U_\infty x)^{1/2} f(\xi)$, the temperature $T_0 = T_\infty g(\xi)$, and an independent variable $\xi = (\rho_\infty U_\infty/(2\mu_\infty x))^{1/2} \int_0^y (\rho_0/\rho_\infty)dy$, the differential equations $(MNf'')' + ff'' = 0$ and $((MN/Pr)g')' + fg' + (\kappa - 1)Ma_\infty^2 MNf''^2 = 0$ were solved for $Pr = 0.7$.

Figures 2-4 show the velocity profiles U_0/U_∞, distributions of temperature T_0/T_∞, density ρ_0/ρ_∞ and viscosity μ_0/μ_∞ in the cases of thermally insulated wall and isothermally conducting walls. Since Sutherland's formula [Eq. (2)] for μ/μ_∞ is a function not only of T/T_∞ but of T_∞, the freestream temperature $T_\infty = 273$ K is given for both the insulated wall and the conducting wall with $T_w/T_\infty = 1.2$, whereas for the case of $T_w/T_\infty = 0.5$ two temperatures, $T_\infty = 273$ K and 1000 K, were considered, because the latter is rather practical for strong cooling. It was ascertained for $T_w/T_\infty = 0.5$ and $Ma_\infty = 5$ that, when the freestream temperature changes from 1000 K to 273 K, variation of the velocity profile in Fig. 2 is less than 10%, and those of the temperature, density, and viscosity in Fig. 4 are less than 5%. The effect of base velocity profile on the present instability was examined for incompressible boundary layers by Görtler,[1] Hämmerlin,[2] and Kobayashi.[10,11] Based on this, it is expected that the slight

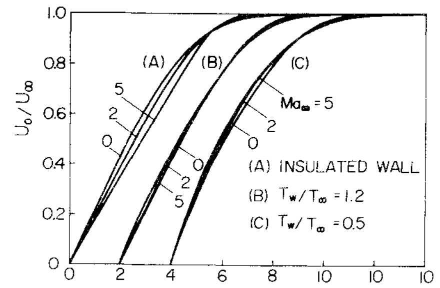

Fig. 2 Base velocity profiles in compressible laminar boundary layers.

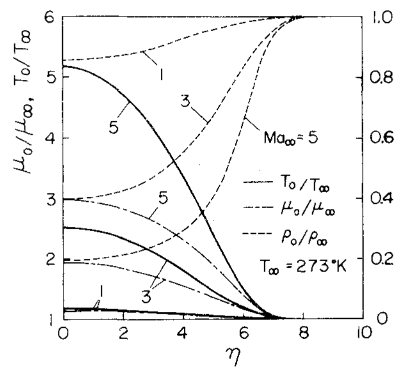

Fig. 3 Distributions of temperature, density, and viscosity in base boundary layers (insulated wall).

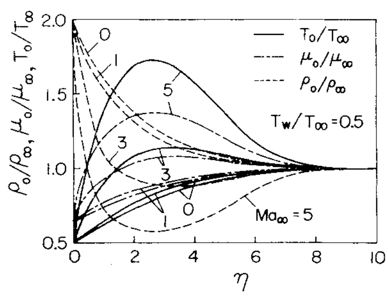

Fig. 4 Distributions of temperature, density, and viscosity in base boundary layers (isothermal wall, $T_w/T_\infty = 0.5$, $T_\infty = 1000$ K).

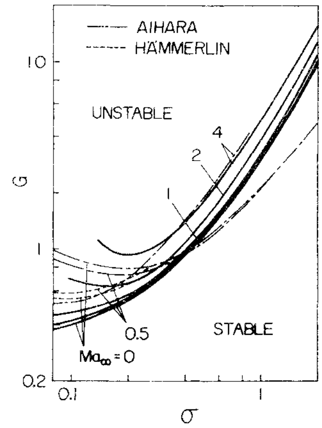

Fig. 5 Neutral stability curves (insulated wall, $T_\infty = 273$ K).

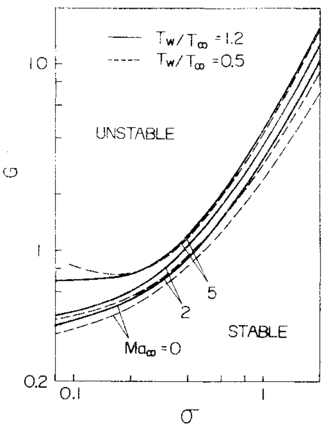

Fig. 6 Neutral stability curves (isothermal wall). $T_\infty = 273$ K for $T_w/T_\infty = 1.2$, $T_\infty = 1000$ K for $T_w/T_\infty = 0.5$.

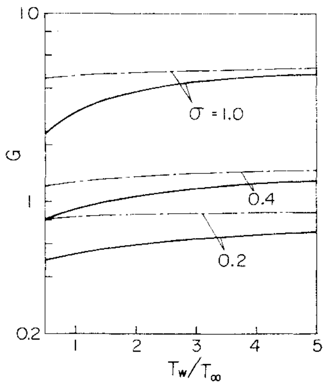

Fig. 7 Variations of G with temperature ratio T_w/T_∞ under isothermal condition ($T_\infty = 273$ K). —— $Ma_\infty = 0$; — — — $Ma_\infty = 5$.

differences in U_0/U_∞ with wide variation in Ma_∞ shown in Fig. 2 will not change G. The fact is in contrast to the critical Reynolds number for Tollmien-Schlichting instability leading to two-dimensional wavy disturbances, in which case the critical Reynolds number depends sensibly on the base velocity profile in boundary layer.[23,24] For the compressible boundary layers, it will be of interest to see how the existence of density gradient and the increase in viscosity seen in Figs. 3 and 4 affect the critical Görtler parameter.

In Figs. 5 and 6, neutral stability curves are shown with the relation of G to the dimensionless wavenumber σ of the longitudinal vortices for different values of Ma_∞. The stable range is below each neutral curve. Figure 5 is for the insulated wall, in which the results by Aihara[20] and Hämmerlin[21] are compared. The present results indicate that the stable region becomes larger with increasing Mach number. Rayleigh's criterion in the case of inviscid flow with curved streamlines states that the flow is unstable when the circulation decreases in the radial direction (in the negative direction of the y axis for the present coordinate system). The criterion is modified for a case with density variation as follows:

$$\frac{1}{\rho_0}\frac{d\rho_0}{dy} + \frac{2}{RU_0}\frac{d(RU_0)}{dy} \quad \begin{matrix} < 0 & \text{stable} \\ > 0 & \text{unstable} \end{matrix} \tag{12}$$

From Eq. (12) it is seen that the positive density gradient, shown in Fig. 3 with broken lines, makes the boundary layer more unstable as compared with the corresponding incompressible boundary layer. On the other hand, the viscosity μ_0 of the gas becomes large in the compressible boundary layer. Since G for the neutral state is defined with the freestream condition (ρ_∞, μ_∞), it increases because the effective viscosity in the boundary layer becomes large. It can, therefore, be said that the stabilization due to the compressibility of the gas, shown in Fig. 5, results because the effect of the increase in viscosity is quantitatively more predominant than the opposite effect of the positive density gradient. The result by Aihara was obtained under an assumption that the viscosity remains unchanged. It follows that G decreases as Ma_∞ increases from 0 to 0.5. Hämmerlin adopted the dependence of the viscosity on the temperature by a relation $\mu_\infty T^{0.78}$ and a constant Prandtl number 0.72, and solved the perturbation equations with power series development of Ma_∞^2. His result by considering the perturbation equations up to terms of Ma_∞^2 indicated that G becomes larger as the Mach number increases. For example, the increase in G for $Ma_\infty = 0$ to 0.5 is 1.7% for $\sigma = 0.2$ from

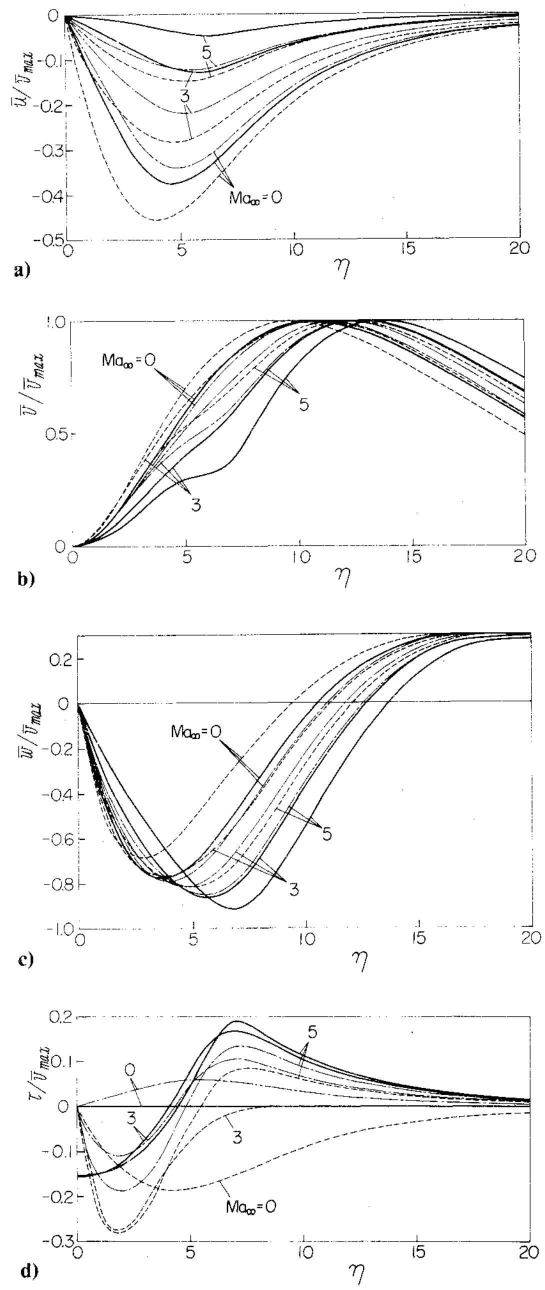

Fig. 8 Perturbation amplitude for velocities $\bar{u}/\bar{v}_{max}$, $\bar{v}/\bar{v}_{max}$, $\bar{w}/\bar{v}_{max}$, and temperature $\bar{\tau}/\bar{v}_{max}$ at onset of longitudinal vortices ($\sigma = 0.2$). —insulated wall; — – —$T_w/T_\infty = 1.2$; ----$T_w/T_\infty = 0.5$ ($T_\infty = 273$ K).

the present results, while the corresponding result by Hämmerlin is 5.4%. It is found that there are differences between the three results for $Ma_\infty = 0$, which occur for the following reasons: Aihara divided the boundary layer thickness into five segments. Hämmerlin adopted another coordinate system, in which the radius R of curvature on streamlines increases exponentially in the y direction. It is known[10] that flows with such streamlines have larger values of G than ones with the constant radius of curvature.

Figure 6 shows neutral curves for two cases of the conducting wall: temperature ratio $T_w/T_\infty = 1.2$ at $T_\infty = 273$ K and $T_w/T_\infty = 0.5$ at $T_\infty = 1000$ K. As shown in Fig. 4, the density profile ρ_0/ρ_∞ has not only positive but negative gradients. The viscosity profile has also two regions of $\mu_0 > \mu_\infty$ and $\mu_0 < \mu_\infty$. It might, therefore, be said that the effects of the variation of the density and viscosity on the present instability are complicated compared to the one for the insulated wall. For the insulated wall, the increment ϵ of the Görtler parameter ($\epsilon = (G - G_0)/G$, G_0 is the value of G at $Ma_\infty = 0$) reaches more than 60% at $Ma_\infty = 5$. The moderate cooling of the solid wall ($T_w/T_\infty = 1.2$) decreases ϵ. The strong cooling ($T_w/T_\infty = 0.5$), however, increases the value of ϵ. The fact seems to come from the stabilizing effect of the negative density gradient shown in Fig. 4. T_∞ itself has also some effects on ϵ. Typically for $Ma_\infty = 5$, $T_w/T_\infty = 0.5$, and $\sigma = 0.2$, ϵ falls from 66% for $T_\infty = 273$ K to 51% for $T_\infty = 1000$ K. To clarify the effect of temperature ratio T_w/T_∞ more precisely, G is given as a function of T_w/T_∞ in the two cases of $Ma_\infty = 0$ and 5 in Fig. 7. It shows that T_w/T_∞ has less effect on the value of G at higher Mach numbers.

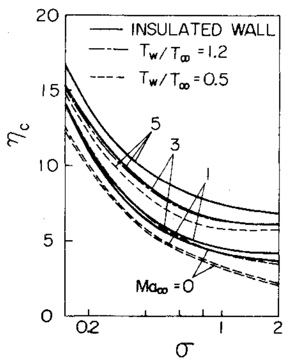

Fig. 9 Variations of center η_c of longitudinal vortices with wavenumber σ and Mach number Ma_∞.

A typical set of the distributions of the perturbation amplitudes for velocities ($\bar{u}$, $\bar{v}$, $\bar{w}$) and the temperature τ at the onset of longitudinal vortices are illustrated in Figs. 8a-d for $\sigma = 0.2$ in the case of insulated wall. The figures show that the locations η of the maximum (v_{max}) and the minima (u_{min}, w_{min}) move away from the wall with increasing Mach number. The location η for $\bar{w} = 0$ in Fig. 8c agrees with the center of the longitudinal vortices, as seen from Eq. (1). Figure 9 gives the location η_c of the vortex center in relation to the wavenumber σ and the Mach number Ma_∞. The center η_c moves closer to the wall as the wavenumber σ increases, but moves away from the wall with increasing value of the Mach number Ma_∞. The cooling of the wall has a remarkable effect on the location η_c.

Conclusions

For compressible laminar boundary layers along concave curved surfaces, it was theoretically considered, over a wide range of Mach numbers, how the compressibility of the gas affects the Taylor-Görtler instability leading to the onset of longitudinal vortices. The results are summarized as follows:

1) The neutral stability curves show that the boundary layer becomes more stable as the Mach number increases. The effect of the Mach number goes the same way as those in the Tollmien-Schlichting instability for supersonic Mach number.[25,26]

2) The present stabilizing effect is considered to result because the stabilization due to the increase in effective viscosity has outweighed the destabilization due to the positive density gradient.

3) For isothermal walls, the temperature ratio T_w/T_∞ has less effect on the critical value of G as Ma_∞ is increased.

4) Distributions of the perturbations (velocities and temperature) at the onset of longitudinal vortices were presented. The center of the vortices moves further from the solid wall as the Mach number becomes larger.

Acknowledgment

Part of this work was supported under the Grant in Aid for Scientific Research from the Ministry of Education, Japan.

References

[1] Görtler, H., "Über eine dreidimensionale Instabilität laminarer Grenzschichten an konkaven Wänden," *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen,* mathematisch-physikalische Klasse, neue Folge, Fachgruppe I, Bd. 2, Nr. 1, 1940.

[2]Hämmerlin, G., "Über das Eigenwertproblem der dreidimensionalen Instabilität laminarer Grenzschichten an konkaven Wänden," *Journal of Rational Mechanics and Analysis,* Vol. 4, March 1955, pp. 279-321.

[3]Witting, H., "Über den Einfluss der Stromlinienkrümmung auf die Stabilität laminarer Strömungen, *Archive for Rational Mechanics and Analysis,"* Vol. 2, 1958, pp. 243-283.

[4]Smith, A.M.O., "On the Growth of Taylor-Görtler Vortices along Highly Concave Wall," *Quarterly Applied Mathematics,* Vol. 13, Oct. 1955, pp. 233-262.

[5]Kobayashi, R., "Stability of Laminar Boundary Layer on a Concave Permeable Wall with Homogeneous Suction," *Journal of Fluid Mechanics,* Vol. 52, March 1972, pp. 269-272.

[6]Kobayashi, R., "Taylor-Görtler Instability of a Boundary Layer with Suction or Blowing," *AIAA Journal,* Vol. 12, March 1974, pp. 394-395.

[7]DiPrima, R.C. and Dunn, D.W., "The Effect of Heating and Cooling on the Stability of the Boundary-Layer Flow of a Liquid over a Curved Surface," *Journal of Aeronautical Sciences,* Vol. 23, Oct. 1956, pp. 913-916.

[8]Kirchgässner, K., "Über den Einfluss des Gleitens bei verdünnten Gasen auf die Entstehung der Taylor-Görtler-Wirbel," *Deutsche Versuchsanstalt für Luftfahrt,* Bericht Nr. 75, Nov. 1958.

[9]Kobayashi, R., "Instabilität laminarer Grenzschichten an konkaven Wänden im transversalen Magnetfeld," *Ingenieur-Archiv,* Bd. 38, Sept. 1969, S. 292-299.

[10]Kobayashi, R., "Stability of Laminar Boundary Layers on Concave Walls in Presence of Magnetic Field," *Bulletin of the Japan Society of Mechanical Engineers,* Vol. 15, July 1972, pp. 822-830.

[11]Kobayashi, R., "Hydromagnetic Instability of Laminar Boundary Layer on a Concave Wall," *Zeitschrift für Angewandte Mathematik und Mechanik,* Bd. 52, July 1972, S. 337-344.

[12]Tani, I., "Production of Longitudinal Vortices in the Boundary Layer along a Concave Wall," *Journal of Geophysical Research,* Vol. 67, July 1962, pp. 3075-3080.

[13]Sandmayr, G., "Über das Auftreten von Längswirbeln in turbulenten Grenzschichten an konkaven Wänden." *Deutsche Luft- und Raumfahrt,* Forschungsbericht 66-41, July 1966.

[14]So, R.M.C. and Mellor, G.L., "Experiment on Turbulent Boundary Layers on a Concave Wall," *Aeronautical Quarterly,* Vol. 26, Feb. 1975, pp. 25-40.

[15]Tokuda, M., "Taylor-Görtler Vortices Expected in the Air Flow on Sea Surface Waves-I," *Journal of Oceanographical Society of Japan,* Vol. 28, Dec. 1972, pp. 14-25.

[16]Liepmann, H.W., "Investigations on Laminar Boundary Layer Stability and Transition on Curved Surfaces," *NACA* Wartime Report W-107, Aug. 1943.

[17]Aihara, Y., "Transition in an Incompressible Boundary Layer along a Concave Wall," *Bulletin of Aeronautical Research Institute,* Tokyo Univ., Vol. 3, Dec. 1962, pp. 195-240.

[18]Wortmann, F.X., "Experimentelle Untersuchungen laminarer Grenzschichten bei instabiler Schichtung," Applied Mechanics, *Proceedings of the 11th International Congress of Applied Mechanics,* Munich, edited by H. Görtler, 1966, pp. 816-825.

[19]Bippes, H. and Görtler, H., "Dreidimensionale Störungen in der Grenzschicht an einer konkaven Wand," *Acta Mechanica,* Bd. 14, 1972, S. 251-267.

[20]Aihara, Y., "Stability of the Compressible Boundary Layer along a Curved Wall under Görtler-Type Disturbances," Aeronautical Research Institute, Tokyo Univ., Rept. No. 362, Feb. 1961, pp. 31-37.

[21]Hämmerlin, G., "Über die Stabilität einer kompressiblen Strömung längs einer konkaven Wand bei verschiedenen Wandtemperaturverhältnissen," *Deutsche Versuchsanstalt für Luftfahrt,* Bericht No. 176, Nov. 1961.

[22]Zakkay, V. and Calarese, W., "An Experimental Investigation of Vortex Generation in a Turbulent Boundary Layer Undergoing Adverse Pressure Gradient," NASA CR-2037, June 1972.

[23]Stuart, J.T., "Hydrodynamic Stability," *Laminar Boundary Layers,* ed. by L. Rosenhead, Oxford Univ. Press, London, 1963, pp. 540-547.

[24]Kobayashi, R., "Hydromagnetische Stabilität laminarer Grenzschicht an einer längsangeströmten ebenen Platte," *Zeitschrift für Angewandte Mathematik und Physik,* Vol. 23, May 1972, pp. 341-352.

[25]Lin, C.C., ed., *"Turbulent Flows and Heat Transfer,"* Princeton University Press, Princeton, 1959, pp. 56-57.

[26]Wazzan, A.R., "Spatial Stability of Tollmien-Schlichting Waves," *Progress in Aerospace Sciences,* Pergamon Press, Vol. 16, 1975, pp. 99-127.